Steenrod operations in the cohomology of exceptional Lie groups

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Abstract

Let G be an exceptional Lie group with a maximal torus T, and let $\mathcal{A}_p$ be the mod–p Steenrod algebra. Based on common properties in the Schubert presentation of the cohomology $H^*(G/T;\mathbb{F}_p)$, we obtain a complete characterization for the $\mathcal{A}_p$–algebra $H^*(G;\mathbb{F}_p)$.

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1 Introduction

Let $\mathcal{A}_p$ be the mod–p Steenrod algebra with $\mathcal{P}^k \in \mathcal{A}_p$, $k \geq 1$, the k^{th} reduced power [SE] and $\delta_p \in \mathcal{A}_p$ the Bockstein operator. If $p = 2$ it is also customary to write Sq^{2k} instead of $\mathcal{P}^k$, Sq^1 in the place of δ_2.

Let G be an 1–connected simple Lie group with $T \subset G$ a maximal torus. Based on common properties in the Schubert presentation of the integral cohomology $H^*(G/T)$, the ring $H^*(G;\mathbb{F})$ was constructed uniformly for all G and $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{F}_p$ in terms of *primary generators* in [DZ$_2$]. In this sequel to [DZ$_2$] we determine the $\mathcal{A}_p$ action on $H^*(G;\mathbb{F}_p)$ with respect to these generators for all exceptional G. We may restrict ourself to the cases where the integral cohomology $H^*(G)$ contain non–trivial p–torsion subgroup, for exactly in these cases the rings $H^*(G;\mathbb{F}_p)$ fail to be primitive generated exterior algebras. The results are requested in [DZ$_2$] to determine the integral cohomology ring $H^*(G)$ for all exceptional G.

The main idea in our approach is to describe the ring $H^*(G;\mathbb{F}_p)$ by the *p–transgressive generators* constructed explicitly from certain polynomials emerging from Schubert presentation of the ring $H^*(G/T;\mathbb{F}_p)$, and to reduce the computation in $H^*(G;\mathbb{F}_p)$ to calculation in these polynomials.

In Theorem 1 below we present the $\mathcal{A}_p$–algebra $H^*(G;\mathbb{F}_p)$ with respect to the *p–transgressive generators* α_i ($\deg \alpha_i = i$), together with the χ^*–images x_{2t} of *the special Schubert classes* y_t on G/T (see in §2 and §3 for their definition). Theorem 2 in §4 decides the relationship between the *p–primary generators* utilized in [DZ$_2$] and the *p–transgressive generators* α_i constructed in this paper. Combining these two results completes the project of this paper, see Remark 3 in §4.

Theorem 1. *Let (G,p) be a pair with G exceptional and $H^*(G)$ containing non–trivial p–torsion. Then*

(1.1) *with respect to the presentations of* $H^*(G;\mathbb{F}_2)$

$$H^*(G_2;\mathbb{F}_2)=\mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle\otimes\Delta_{\mathbb{F}_2}(\alpha_3,\alpha_5);$$
$$H^*(F_4;\mathbb{F}_2)=\mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle\otimes\Delta_{\mathbb{F}_2}(\alpha_3,\alpha_5,\alpha_{15},\alpha_{23});$$
$$H^*(E_6;\mathbb{F}_2)=\mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle\otimes\Delta_{\mathbb{F}_2}(\alpha_3,\alpha_5,\alpha_9,\alpha_{15},\alpha_{17},\alpha_{23});$$
$$H^*(E_7;\mathbb{F}_2)=\frac{\mathbb{F}_2[x_6,x_{10},x_{18}]}{\left\langle x_6^2,x_{10}^2,x_{18}^2\right\rangle}\otimes\Delta_{\mathbb{F}_2}(\alpha_3,\alpha_5,\alpha_9,\alpha_{15},\alpha_{17},\alpha_{23},\alpha_{27});$$
$$H^*(E_8;\mathbb{F}_2)=\frac{\mathbb{F}_2[x_6,x_{10},x_{18},x_{30}]}{\left\langle x_6^8,x_{10}^4,x_{18}^2,x_{30}^2\right\rangle}\otimes\Delta_{\mathbb{F}_2}(\alpha_3,\alpha_5,\alpha_9,\alpha_{15},\alpha_{17},\alpha_{23},\alpha_{27},\alpha_{29}),$$

all nontrivial actions of $\mathcal{A}_2$ on $H^*(G;\mathbb{F}_2)$ are given by

$\delta_2(\alpha_5)=x_6$ *in* G_2,F_4,E_6,E_7,E_8;

$\delta_2(\alpha_{2r-1})=x_{2r}$, $r=5,9$;

$\delta_2(\alpha_{15})=x_6x_{10}$; $\quad\delta_2(\alpha_{27})=x_{10}x_{18}$ *in* E_7,E_8

$\delta_2(\alpha_{23})=x_6x_{18}$ *in* E_7;

$\delta_2(\alpha_{23})=x_6x_{18}+x_6^4$; $\quad\delta_2(\alpha_{29})=x_{30}+x_6^2x_{18}$ *in* E_8,

$\mathcal{P}^1\alpha_3=\alpha_5$ *in* G_2,F_4,E_6,E_7,E_8;

$\mathcal{P}^4\alpha_{15}=\alpha_{23}$ *in* F_4,E_6,E_7,E_8;

$\mathcal{P}^2\alpha_5=\alpha_9$; $\mathcal{P}^4\alpha_9=\mathcal{P}^1\alpha_{15}=\alpha_{17}$ *in* E_6,E_7,E_8;

$\mathcal{P}^2\alpha_{23}=\alpha_{27}$ *in* E_7,E_8;

$\mathcal{P}^3\alpha_{23}=\mathcal{P}^1\alpha_{27}=\alpha_{29}$ *in* E_8.

(1.2) *with respect to the presentations of* $H^*(G;\mathbb{F}_3)$

$$H^*(F_4;\mathbb{F}_3)=\mathbb{F}_3[x_8]/\left\langle x_8^3\right\rangle\otimes\Lambda_{\mathbb{F}_3}(\alpha_3,\alpha_7,\alpha_{11},\alpha_{15});$$
$$H^*(E_6;\mathbb{F}_3)=\mathbb{F}_3[x_8]/\left\langle x_8^3\right\rangle\otimes\Lambda_{\mathbb{F}_3}(\alpha_3,\alpha_7,\alpha_9,\alpha_{11},\alpha_{15},\alpha_{17});$$
$$H^*(E_7;\mathbb{F}_3)=\mathbb{F}_3[x_8]/\left\langle x_8^3\right\rangle\otimes\Lambda_{\mathbb{F}_3}(\alpha_3,\alpha_7,\alpha_{11},\alpha_{15},\alpha_{19},\alpha_{27},\alpha_{35});$$
$$H^*(E_8;\mathbb{F}_3)=\mathbb{F}_3[x_8,x_{20}]/\left\langle x_8^3,x_{20}^3\right\rangle\otimes\Lambda_{\mathbb{F}_3}(\alpha_3,\alpha_7,\alpha_{15},\alpha_{19},\alpha_{27},\alpha_{35},\alpha_{39},\alpha_{47})$$

all nontrivial actions of $\mathcal{A}_3$ on $H^*(G;\mathbb{F}_3)$ are given by

$\delta_3(\alpha_7)=-x_8$; $\quad\delta_3(\alpha_{15})=x_8^2$ *in* F_4,E_6,E_7,E_8;

$\delta_3(\alpha_{19})=-x_{20}$; $\quad\delta_3(\alpha_{27})=x_8x_{20}$; $\quad\delta_3(\alpha_{35})=-x_8^2x_{20}$;

$\delta_3(\alpha_{39})=-x_{20}^2$; $\quad\delta_3(\alpha_{47})=-x_8x_{20}^2$ *in* E_8

$\mathcal{P}^1\alpha_3=\alpha_7$ *in* F_4,E_6,E_7,E_8;

$\mathcal{P}^1\alpha_{11}=\alpha_{15}$ *in* F_4,E_6,E_7;

$\mathcal{P}^1\alpha_{15}=\mathcal{P}^3\alpha_7=\alpha_{19}$; $\mathcal{P}^3\alpha_{15}=\alpha_{27}$ *in* E_7,E_8;

$\mathcal{P}^2\alpha_{11}=-\alpha_{19}$ *in* E_7;

$\mathcal{P}^1\alpha_{35}=\mathcal{P}^3\alpha_{27}=\alpha_{39}$; $\mathcal{P}^3\alpha_{35}=\alpha_{47}$ *in* E_8.

(1.3) *with respect to the presentation of* $H^*(E_8;\mathbb{F}_5)$

$$\mathbb{F}_5[x_{12}]/\left\langle x_{12}^5\right\rangle\otimes\Lambda(\alpha_3,\alpha_{11},\alpha_{15},\alpha_{23},\alpha_{27},\alpha_{35},\alpha_{39},\alpha_{47})$$

all nontrivial actions of $\mathcal{A}_5$ on $H^*(E_8;\mathbb{F}_5)$ are given by

$$\delta_5(\alpha_{11}) = -x_{12};\ \delta_5(\alpha_{23}) = -x_{12}^2;\ \delta_5(\alpha_{35}) = x_{12}^3;\ \delta_5(\alpha_{47}) = 2x_{12}^4$$
$$\mathcal{P}^1\alpha_i = \alpha_{i+8},\ i = 3, 15, 27, 39.$$

In the classical descriptions of $H^*(E_7;\mathbb{F}_2)$ and $H^*(E_8;\mathbb{F}_2)$ in [Ar, AS, T, Ko$_1$, KN, Ka, Mi] the generators were specified mainly up to the degrees and the action of $Sq^1 = \delta_2$ on the generators in degrees $15, 23, 27$ was absent. With respect to our explicit construction, results in (1.1) constitutes a complete characterization of $H^*(G;\mathbb{F}_2)$ as an algebra over $\mathcal{A}_2$, see Corollary 1 and Remark 1 in §4.

In [KM] Kono and Mimura largely determined the $\mathcal{A}_3$ action on $H^*(E_7;\mathbb{F}_3)$ and $H^*(E_8;\mathbb{F}_3)$ with respect also to a set of transgressive generators, except an indeterminacy $\epsilon = \pm 1$ occurred in their expressions of $\mathcal{P}^1 e_{11}$, $\mathcal{P}^2 e_{11}, \mathcal{P}^1 e_{15}$ in E_7, and of $\mathcal{P}^1 e_{15}$, $\mathcal{P}^1 e_{35}$ in E_8. Again, with respect to our explicit construction these ambiguities are clarified in (1.2).

Results in (1.3) essentially agrees with the calculation by Kono in [Ko$_2$, Theorem 5.15], whose generators $x_3, x_{15}, x_{27}, x_{39}$ are also transgressive, and correspond to our $\alpha_3, 2\alpha_{15}, 3\alpha_{27}, 2\alpha_{39}$ respectively.

Since 1950's there have been extensive works concerning the $\mathcal{A}_p$–algebra $H^*(G;\mathbb{F}_p)$, for references see Kane [Ka], Lin [L]. However, the classical results fail to imply Theorem 1 because traditionally $H^*(G;\mathbb{F}_p)$ was presented by generators specified mainly by their degrees regardless of the crucial fact that the algebra $H^*(G;\mathbb{F}_p)$ may admits many sets of generators subject to a given presentation and in contrast, the algebra $H^*(G;F_p)$ in Theorem 1 is presented by *explicitly constructed generators*. It is also for this reason that Theorem 1 plays a role in determining the integral cohomology $H^*(G)$ while the classical descriptions fall short of this advantage (e.g. discussion prior to Corollary 2 in §4.3). In addition, our approach applies uniformly to all G and p while, historically, the calculation were performed case by case depending on G and p ([L$_1$]).

From 1970's there have been important and deep approaches to the $\mathcal{A}_p$–algebras $H^*(G;\mathbb{F}_p)$ from much more general point of view. The theory of James Lin and Richard Kane on finite H-spaces [Ka] may be applied to determine the $\mathcal{A}_p$–algebra $H^*(G;\mathbb{F}_p)$ from the rational cohomology $H^*(G;\mathbb{Q})$ [Ka, KLN]. There are also extensive results of Kono, Hara, Hamanaka, Lin, Nishimura, Kozima using the adjoint action to determine the $\mathcal{A}_p$ action, for references see Lin [L]. It would be interesting to see that these methods can be so developed as to be functional uniformly to all G and p [L$_1$].

2 Schubert presentation of $H^*(G/T;\mathbb{F}_p)$

For a Lie group G with a maximal torus T consider the fibration

(2.1) $G/T \overset{\psi}{\hookrightarrow} BT \overset{\pi}{\to} BG$

induced by the inclusion $T \subset G$, where BT (resp. BG) is the classifying space of T (resp. G). Since $H^{odd}(BT) = H^{odd}(G/T) = 0$, the cohomology exact sequence of the pair $(BT, G/T)$ in the $\mathbb{F}_p$ coefficients contains the section

(2.2) $0 \to H^{even}(BT, G/T;\mathbb{F}_p) \overset{j}{\to} H^*(BT;\mathbb{F}_p) \overset{\psi_p^*}{\to} H^*(G/T;\mathbb{F}_p)$

where as is classical $H^*(BT;\mathbb{F}_p)$ can be identified with the free polynomial ring $\mathbb{F}_p[\omega_1,\cdots,\omega_n]$ in a set of fundamental dominant weights $\omega_1,\cdots,\omega_n \in H^2(BT;\mathbb{F}_p)$ of G, and where the ring map ψ_p^* induced by the fiber inclusion ψ is called the *Borel's characteristic map* in characteristic p [BH, B$_1$].

It is well known from Borel [B$_1$] that if the integral cohomology $H^*(G)$ is free of p–torsion, then ψ_p^* is surjective and induces an isomorphism

$$H^*(G/T;\mathbb{F}_p) = H^*(BT;\mathbb{F}_p)/\left\langle H^+(BT;\mathbb{F}_p)^{W(G)}\right\rangle$$

where $\left\langle H^+(BT;\mathbb{F}_p)^{W(G)}\right\rangle$ is the ideal in $H^*(BT;\mathbb{F}_p)$ generated by Weyl invariants in positive degrees (see Demazure [D] for another proof of this fact). Without any restriction on the torsion subgroup of $H^*(G)$ Lemma 1 below extends this classical result.

For simplicity, we make no difference in notation between a polynomial $\theta \in H^*(BT;\mathbb{F}_p)$ and its ψ_p^*–image in $H^*(G/T;\mathbb{F}_p)$. Given a subset $\{f_1,\ldots,f_m\}$ in a ring write $\langle f_1,\ldots,f_m\rangle$ for the ideal generated by $f_1,\ldots,f_m$.

Lemma 1 ([DZ$_1$, Proposition 3]). *For each* 1*–connected Lie group* G *with rank* n *and and a prime* p*, there exist*

a set $\{\theta_{s_1},\cdots,\theta_{s_n}\} \subset H^*(BT;\mathbb{F}_p)$ *of* n *polynomials; and*

a set $\{y_{t_1},\cdots,y_{t_k}\} \subset H^*(G/T;\mathbb{F}_p)$ *of Schubert classes on* G/T

with $\deg\theta_s = 2s$, $\deg y_t = 2t > 2$, *so that*

i) $\ker\psi_p^* = \langle\theta_{s_1},\cdots,\theta_{s_n}\rangle$;

ii) $H^*(G/T;\mathbb{F}_p) = \mathbb{F}_p[\omega_1,\ldots,\omega_n,y_t]/\left\langle \theta_s, y_t^{k_t}+\beta_t\right\rangle_{s\in r(G,p),\ t\in e(G,p)}$;

iii) the three sets $r(G,p)$, $e(G,p)$ *and* $\{k_t\}_{t\in e(G,p)}$ *of integers are subject to the constraints*

$$e(G,p) \subset r(G,p); \qquad \dim G = \sum_{s\in r(G,p)}(2s-1) + \sum_{t\in e(G,p)} 2(k_t-1)t,$$

where $r(G,p) = \{s_1,\cdots,s_n\}$, $e(G,p) = \{t_1,\cdots,t_k\}$ *and* $\beta_t \in \langle\omega_1,\ldots,\omega_n\rangle$.$\square$

Since the set $\{\omega_1,\cdots,\omega_n\}$ of fundamental dominant weights consists of all Schubert classes on G/T with cohomology degree 2, ii) of Lemma 1 describes the ring $H^*(G/T;\mathbb{F}_p)$ by certain Schubert classes on G/T and therefore, will be called *a Schubert presentation* of $H^*(G/T;\mathbb{F}_p)$. In addition to $\{\omega_1,\cdots,\omega_n\}$ elements in the set $\{y_t\}_{t\in e(G,p)}$ will be called the *p–special Schubert classes* on G/T. For each exceptional G and prime p, a set of p–special Schubert classes on G/T has been determined in [DZ$_1$], and is specified by their Weyl coordinates in the table below:

y_i	G_2/T	F_4/T	$E_n/T,\ n=6,7,8$	p
y_3	$\sigma_{[1,2,1]}$	$\sigma_{[3,2,1]}$	$\sigma_{[5,4,2]}$, $n=6,7,8$	2
y_4		$\sigma_{[4,3,2,1]}$	$\sigma_{[6,5,4,2]}$, $n=6,7,8$	3
y_5			$\sigma_{[7,6,5,4,2]}$, $n=7,8$	2
y_6			$\sigma_{[1,3,6,5,4,2]}$, $n=8$	5
y_9			$\sigma_{[1,5,4,3,7,6,5,4,2]}$, $n=7,8$	2
y_{10}			$\sigma_{[1,6,5,4,3,7,6,5,4,2]}$, $n=8$	3
y_{15}			$\sigma_{[5,4,2,3,1,6,5,4,3,8,7,6,5,4,2]}$, $n=8$	2

The p–special Schubert classes on G/T and their abbreviations

In view of i) of Lemma 1 we shall call $\{\theta_s\}_{s\in r(G,p)}$ a set of *generating polynomials* for $\ker\psi_p^*$. These polynomials have been emphasized by Kač [K] as *a regular sequence of homogeneous generators* for $\ker\psi_p^*$; notified by Ishitoya, Kono and Toda [IKT, Theorem 1.1] as the *transgressive imagines* of a set of transgressive generators on $H^*(G;\mathbb{F}_p)$. However, it is in the context of [DZ$_1$, §6] that concrete presentation of a set of such polynomials is available for every exceptional G and prime p.

Assume in the remainder of this section that (G,p) is a pair with G exceptional and $H^*(G)$ containing non–trivial p–torsion. Explicitly, we shall have

$p=2$: $G=G_2,F_4,E_6,E_7,E_8$;
$p=3$: $G=F_4,E_6,E_7,E_8$; and
$p=5$: $G=E_8$.

For these cases a set of *generating polynomials* for $\ker\psi_p^*$ are presented in Propositions 2–4 in §5.2; and the sets $r(G,p)$, $e(G,p)$ and $\{k_t\}_{t\in e(G,p)}$ of integers appearing in Lemma 1 are tabulated below, where $e(G,p)$ is given as the subset of $r(G,p)$ whose elements are underlined:

(G,p)	$e(G,p)\subset r(G,p)$	$\{k_t\}_{t\in e(G,p)}$
$(G_2,2)$	$\{2,\underline{3}\}$	$\{2\}$
$(F_4,2)$	$\{2,\underline{3},8,12\}$	$\{2\}$
$(E_6,2)$	$\{2,\underline{3},5,8,9,12\}$	$\{2\}$
$(E_7,2)$	$\{2,\underline{3},\underline{5},8,\underline{9},12,14\}$	$\{2,2,2\}$
$(E_8,2)$	$\{2,\underline{3},\underline{5},8,\underline{9},12,14,\underline{15}\}$	$\{8,4,2,2\}$
$(F_4,3)$	$\{2,\underline{4},6,8\}$	$\{3\}$
$(E_6,3)$	$\{2,\underline{4},5,6,8,9\}$	$\{3\}$
$(E_7,3)$	$\{2,\underline{4},6,8,10,14,18\}$	$\{3\}$
$(E_8,3)$	$\{2,\underline{4},8,\underline{10},14,18,20,24\}$	$\{3,3\}$
$(E_8,5)$	$\{2,\underline{6},8,12,14,18,20,24\}$	$\{5\}$

.

Combining (2.2) with i) of Lemma 1 we get the short exact sequence

$$(2.3)\quad 0\to H^{even}(BT,G/T;\mathbb{F}_p)\xrightarrow{j}H^*(BT;\mathbb{F}_p)\xrightarrow{\psi_p^*}\frac{H^*(BT;\mathbb{F}_p)}{\langle\theta_s\rangle_{s\in r(G,p)}}\to 0.$$

It implies that j identifies $H^{even}(BT,G/T;\mathbb{F}_p)$ with $\ker\psi_p^*=\langle\theta_i\rangle_{i\in r(G,p)}$. In particular, $\{\theta_i\}_{i\in r(G,p)}\subset H^*(BT,G/T;\mathbb{F}_p)$. It follows that, for any pair $\{s,t\}\subset r(G,p)$ with $t=s+k(p-1)$, there exists a unique $b_{s,t}\in\mathbb{F}_p$ so that a relation of the form

$$(2.4)\quad \mathcal{P}^k(\theta_s)=b_{s,t}\theta_t+\tau_t \text{ with } \tau_t\in\langle\theta_s\rangle_{s\in r(G,p),s<t}$$

holds in $H^*(BT,G/T;\mathbb{F}_p)$ (resp. in $H^*(BT;\mathbb{F}_p)$ via the injection j). Based on the concrete presentation of $\{\theta_i\}_{i\in r(G,p)}$ in §5.2 the next result is proved in §5.3 by direct computation in the simpler ring $H^*(BT;\mathbb{F}_p)$:

Lemma 2. *With respect to the degree set $r(G,p)$ of the generating polynomials for $\ker\psi_p^*$ (§5.2) specified in the table, all non–zero $b_{s,t}$ in (2.4) are given by*

$p=2:$ $b_{2,3}=1$ *for* G_2,F_4,E_6,E_7,E_8;

$b_{8,12}=1$ *for* F_4,E_6,E_7,E_8;

$b_{3,5}=b_{5,9}=b_{8,9}=1$ *for* E_6,E_7,E_8;

$b_{12,14} = 1$ *for* E_7, E_8;

$b_{12,15} = b_{14,15} = 1$ *for* E_8.

$p = 3:$ $b_{2,4} = 1$ *for* F_4, E_6, E_7, E_8;

$b_{6,8} = 1$ *for* F_4, E_6, E_7;

$b_{4,10} = b_{8,14} = b_{8,10} = 1$ *for* E_7, E_8;

$b_{6,10} = -1$ *for* E_7;

$b_{18,20} = b_{14,20} = b_{18,24} = 1$ *for* E_8;

$p = 5:$ $b_{k,k+4} = 1$ *for* $G = E_8$ *and* $k = 2, 8, 14, 20$.

3 $H^*(G;\mathbb{F}_p)$ as a module over $\mathcal{A}_p$

In this section we construct $H^*(G;\mathbb{F}_p)$ from the presentation of $H^*(G/T;\mathbb{F}_p)$ in ii) of Lemma 2, and specify the $\mathcal{P}^k$ action on $H^*(G;\mathbb{F}_p)$ by $b_{s,t} \in \mathbb{F}_p$ in (2.4).

The pull back of the universal T–bundle $E_T \to BT$ via the fiber inclusion ψ in (2.1) gives rise to the principle T–bundle

(3.1) $T \to G \overset{\chi}{\to} G/T$.

Since G/T is 1–connected, the *Borel transgression* $\tau : H^1(T;\mathbb{F}_p) \to H^2(G/T;\mathbb{F}_p)$ defines a basis $\{t_i\}_{1\leq i\leq n}$ of $H^1(T;\mathbb{F}_p)$ by $\tau(t_i) = \omega_i$. Consequently, $H^*(T;\mathbb{F}_p) = \Lambda^*_{\mathbb{F}_p}(t_1,\ldots,t_n)$, and in the Leray–Serre spectral sequence $\{E_r^{*,*}(G;\mathbb{F}_p), d_r\}$ of (3.1) one has

(3.2) $E_2^{s,t}(G;\mathbb{F}_p) = H^s(G/T) \otimes \Lambda^t_{\mathbb{F}_p}(t_1,\ldots,t_n)$;

(3.3) the differential $d_2 : E_2^{s,t}(G;\mathbb{F}_p) \to E_2^{s+2,t-1}(G;\mathbb{F}_p)$ is given by

$$d_2(x \otimes t_k) = x\omega_k \otimes 1,\ x \in H^s(G/T;\mathbb{F}_p),\ 1 \leq k \leq n.$$

Over $\mathbb{F}_p$ the subring $H^+(BT;\mathbb{F}_p)$ has the canonical additive basis $\{\omega_1^{b_1}\cdots\omega_n^{b_n} \mid b_i \geq 0, \sum b_i \geq 1\}$. Consider the $\mathbb{F}_p$–linear map

(3.4) $\mathcal{D} : H^+(BT;\mathbb{F}_p) \to E_2^{*,1}(G;\mathbb{F}_p) = H^*(G/T;\mathbb{F}_p) \otimes \Lambda^1_{\mathbb{F}}$

by $\mathcal{D}(\omega_1^{b_1}\cdots\omega_n^{b_n}) = \omega_1^{b_1}\cdots\omega_s^{b_s-1}\cdots\omega_n^{b_n} \otimes t_s$, where $s \in \{1,\cdots,n\}$ is the least one with $b_s \geq 1$. Immediate but useful properties of the map $\mathcal{D}$ are:

Lemma 3. *Let* $\beta_1, \beta_2 \in H^+(BT;\mathbb{F}_p)$, *and write* $[\theta] \in E_3^{s,t}(G;\mathbb{F}_p)$ *for the cohomology class of a* d_2*–cocycle* $\theta \in E_2^{s,t}(G;\mathbb{F}_p)$. *Then*

i) $D(\ker\psi_p^*) \subset \ker d_2$; *ii)* $D(\beta_1\beta_2) - \beta_1 D(\beta_2) \in \operatorname{Im} d_2$.

In particular, $[\mathcal{D}(\beta_1\beta_2)] = 0$ *if either* β_1 *or* $\beta_2 \in \ker\psi_p^*$.

Proof. i) is shown by $d_2(\mathcal{D}(\theta)) = \theta = 0$ in $H^*(G/T;\mathbb{F}_p)$ for all $\theta \in \ker\psi_p^*$. For ii) it suffices to consider the cases where β_1, β_2 are monomials in $\omega_1,\cdots,\omega_n$, and the result comes directly from the definition of $\mathcal{D}$.□

By i) of Lemma 3, $\mathcal{D}$ assigns each generating polynomial θ_s an element

(3.5) $\alpha_{2s-1} =: [\mathcal{D}(\theta_s)] \in E_3^{2s-2,1}(G;\mathbb{F}_p)$.

Since $E_2^{s,t}(G;\mathbb{F}) = 0$ for s odd, one has the *canonical* monomorphism

$$E_3^{2k,1}(G;\mathbb{F}_p) = E_\infty^{2k,1}(G;\mathbb{F}_p) = \mathcal{F}^{2k}H^{2k+1}(G;\mathbb{F}_p) \subset H^{2k+1}(G;\mathbb{F}_p)$$

which interprets directly α_{2s-1} as a cohomology class of G, where $\mathcal{F}$ is the filtration on $H^*(G;\mathbb{F}_p)$ induced from χ. Furthermore, by Lemma 3 if we write $\mathcal{T}$ for the subspace of $H^*(G;\mathbb{F}_p)$ spanned by the set $\{\alpha_{2s-1}\}_{s\in r(G,p)}$, the map $\mathcal{D}$ in (3.4) restricts to a surjection

(3.6) $[\mathcal{D}] : \ker\psi_p^* = H^+(BT, G/T;\mathbb{F}_p) \to \mathcal{T} \subset H^{odd}(G;\mathbb{F}_p)$.

Let $\{y_t\}_{t\in e(G,p)}$ be the set of p–special Schubert classes on G/T and put $x_{2t} := \chi^* y_t \in H^{2t}(G;\mathbb{F}_p)$. Denote by $\Delta(\alpha_{2s-1})_{s\in r(G,p)}$ the $\mathbb{F}_p$–module in the simple system $\{\alpha_{2s-1}\}_{s\in r(G,p)}$ of generators. We formulate $H^*(G;\mathbb{F}_p)$ from the presentation of $H^*(G/T;\mathbb{F}_p)$ in ii) of Lemma 1, and specify $\mathcal{P}^k$ action on $H^*(G;\mathbb{F}_p)$ in terms of the coefficients $b_{s,t} \in \mathbb{F}_p$ in (2.4).

Lemma 4. *The inclusion $\{\alpha_{2s-1}\}_{s\in r(G,p)}$, $\{x_{2t}\}_{t\in e(G,p)} \subset H^*(G;\mathbb{F}_p)$ induces an isomorphism of $\mathbb{F}_p$–modules*

i) $H^*(G;\mathbb{F}_p) = \mathbb{F}_p[x_{2t}]/\left\langle x_{2t}^{k_t}\right\rangle_{t\in e(G,p)} \otimes \Delta(\alpha_{2s-1})_{s\in r(G,p)}$.

Moreover, $\mathcal{T}$ is an invariant subspace of all P^k and

ii) (2.4) implies that $P^k\alpha_{2s-1} = b_{s,t}\alpha_{2t-1}$.

Proof. Assertions i) may be considered as known, see Kač [K, Theorem 3] or Ishitoya, Kono and Toda [IKT; Theorem 1.1]. We outline a proof for it because certain ideas in the process are required by showing ii).

From ii) of Lemma 1 and (3.3) we find that

$$E_3^{*,0} = \operatorname{Im}\chi^* = \mathbb{F}_p[x_{2t}]/\left\langle x_{2t}^{k_t}\right\rangle_{t\in e(G,p)} \subset H^*(G;\mathbb{F}_p).$$

The same method as that used in establishing [DZ$_2$, Lemma 3.2] is applicable to show that $E_3^{*,1}$ is spanned by $\{\alpha_{2s-1}\}_{s\in r(G,p)}$ (as a module over $E_3^{*,0}$). Further, since $E_3^{*,*}$ is generated multiplicatively by $E_3^{*,0}$ and $E_3^{*,1}$ [K, S], and since

$$E_3^{\dim G-n,n} = E_2^{\dim G-n,n} = \mathbb{F}_p$$

(for $E_2^{\dim G-n-2,n+1} = E_2^{\dim G-n+2,n-1} = 0$), we get from iii) of Lemma 1 that

$$E_3^{*,*} = \mathbb{F}_p[x_{2t}]/\left\langle x_{2t}^{k_t}\right\rangle_{t\in e(G,p)} \otimes \Delta(\alpha_{2s-1})_{s\in r(G,p)}.$$

The proof for i) is completed by $E_3^{*,*} = E_\infty^{*,*} = H^*(G;\mathbb{F}_p)$, where the first equality comes from $E_3^{*,0}$, $E_3^{*,1} \subset H^*(G;\mathbb{F}_p)$.

Turning to ii) the short exact sequence (2.3) induces the exact sequence of complexes

$$0 \to H^*(BT, G/T;\mathbb{F}_p) \otimes \Lambda^* \to H^*(BT;\mathbb{F}_p) \otimes \Lambda^* \to \mathcal{A}_2^{*,*} \to 0,$$

in which $\Lambda^* = \Lambda^*_{\mathbb{F}_p}(t_1, \ldots, t_n)$, $\mathcal{A}_2^{*,*} = \frac{H^*(BT;\mathbb{F}_p)}{\langle\theta_i\rangle_{i\in r(G,p)}} \otimes \Lambda^*$ and

$$H^*(BT, G/T;\mathbb{F}_p) \otimes \Lambda^* = E_2^{*,*}(E_T, G;\mathbb{F}_p);$$
$$H^*(BT;\mathbb{F}_p) \otimes \Lambda^* = E_2^{*,*}(E_T;\mathbb{F}_p),$$

where E_T is the total space of the universal T–bundle on BT. It is clear that $\mathcal{A}_2^{*,*}$ is a subcomplex of $E_2^{*,*}(G;\mathbb{F}_p)$ with

$$\mathcal{A}_3^{*,1} = \mathcal{T} \text{ and } \mathcal{A}_3^{*,*} = \Delta(\alpha_{2i-1})_{i\in r(G,p)} \subset H^*(G;\mathbb{F}_p),$$

Since $E_3^{*,*}(E_T;\mathbb{F}_p) = 0$ the connecting homomorphisms in cohomologies give rise to the isomorphisms

$$\beta : \mathcal{A}_3^{*,1} = \mathcal{T} \to E_3^{*,0}(E_T, G;\mathbb{F}_p);$$
$$\beta' : H^{odd}(G;\mathbb{F}_p) \to H^{even}(E_T, G;\mathbb{F}_p)$$

that fit in the commutative diagrams

$$(3.7)\qquad \begin{array}{cccccc} 0 \to & H^{odd}(G;\mathbb{F}_p) & \overset{\beta'}{\underset{\cong}{\to}} & H^{even}(E_T, G;\mathbb{F}_p) & \to 0 & \\ & \cup & & \cup\ \kappa & & \\ 0 \to & \mathcal{T} & \overset{\beta}{\underset{\cong}{\to}} & E_3^{even,0}(E_T, G;\mathbb{F}_p) & \to 0 & , \\ & & \nwarrow [\mathcal{D}] & \uparrow \chi^* & & \\ & & & H^{even}(BT, G/T;\mathbb{F}_p) & & \end{array}$$

where the inclusion κ identifies $E_3^{even,0}(E_T, G;\mathbb{F}_p)$ with the subring

$$\operatorname{Im} \chi^*[H^{even}(BT, G/T;\mathbb{F}_p) \to H^{even}(E_T, G;\mathbb{F}_p)].$$

(by a standard property of Leray–Serre spectral sequence). Since $[\mathcal{D}] = (\beta')^{-1} \circ \chi^*$ by (3.7) and since β' and χ^* commute with $\mathcal{P}^k$, we obtain ii).$\square$

In the context of [IKT; Theorem 1.1] the class $\alpha_{2s-1} \in H^{odd}(G;\mathbb{F}_p)$ are called *transgressive* with *transgressive image* θ_s, $s \in r(G,p)$. So it is appropriate to introduced the next definition (in view of i) of Lemma 4).

Definition 1. Elements in the set $\{\alpha_{2s-1}\}_{s\in r(G,p)}$ are called *p–transgressive generators* on $H^*(G;\mathbb{F}_p)$.$\square$

4 Main results

Assume in this section that G is exceptional with $H^*(G)$ containing non–trivial p–torsion. Let $\{\alpha_{2s-1}\}_{s\in r(G,p)}$ be the set of p–transgressive generators on $H^*(G;\mathbb{F}_p)$ with $\alpha_{2s-1} =: [\mathcal{D}(\theta_s)]$ ((3.5)), where θ_s is given as that in Proposition 2–4 of §5.

In §4.1 we determine the relationship between p–primary generators introduced in [DZ$_2$, Definition 2.3] and the p–transgressive generators on $H^*(G;F_p)$

defined above. Combining Lemma 4, Lemma 2 and Theorem 2, a proof of Theorem 1 is given in §4.2. Results in Theorems 1 and 2 suffice to determine the structure of $H^*(G;\mathbb{F}_p)$ as an $\mathcal{A}_p$–module with respect to the p–primary generators. This is explained in §4.3.

4.1. Relationship between the p–primary and the p–transgressive generators on $H^*(G;\mathbb{F}_p)$. Let $\mathcal{O}_{G,\mathbb{F}_p} = \left\{\xi_{2s-1}\right\}_{s\in r(G,p)} \subset E_3^{*,1}(G;\mathbb{F}_p)$ be the set of p–primary generators introduced in [Definition 2.3, DZ$_2$]. Since $E_3^{*,1}(G,\mathbb{F}_p)$ is a $E_3^{*,0}$ module with basis $\{\alpha_{2s-1}\}_{s\in r(G,p)}$ by the proof of Lemma 4, each $\xi_{2s-1} \in \mathcal{O}_{G,\mathbb{F}_p}$ has an expression in the form

$$(4.1)\quad \xi_{2s-1} = \sum_{i\in r(G,p), i\leq s} g_i\alpha_{2i-1} \text{ with } g_i \in E_3^{*,0} = \mathbb{F}_p[x_{2t}]/\left\langle x_{2t}^{k_t}\right\rangle_{t\in e(G,p)}.$$

Theorem 2. *We have $\xi_{2s-1} = \alpha_{2s-1}$ with the following exceptions*

i) *for $p = 2$ and in E_7, E_8 :*

$\xi_{15} = \alpha_{15} + x_6\alpha_9$; $\quad \xi_{27} = \alpha_{27} + x_{10}\alpha_{17}$ *in* E_7, E_8,
$\xi_{23} = \alpha_{23} + x_6\alpha_{17}$ *in* E_7;
$\xi_{23} = \alpha_{23} + x_6\alpha_{17} + x_6^3\alpha_5$; $\quad \xi_{29} = \alpha_{29} + x_6^2\alpha_{17}$ *in* E_8.

ii) *for $p = 3$*

$\xi_{15} = \alpha_{15} - x_8\alpha_7$ *in* F_4, E_6, E_7, E_8;
$\xi_{35} = \alpha_{35} + x_8\alpha_{27}$ *in* E_7, E_8;
$\xi_{27} = \alpha_{27} + x_8\alpha_{19}$; $\quad \xi_{39} = \alpha_{39} - x_{20}\alpha_{19}$; $\quad \xi_{47} = \alpha_{47} - x_8\alpha_{39}$ *in* E_8.

iii) *for $p = 5$ and in E_8:*

$$\xi_s = \begin{cases} 3\alpha_{15} \text{ for } s = 15; \\ 3\alpha_{23} + 2x_{12}\alpha_{11} \text{ for } s = 23; \\ -\alpha_{35} - x_{12}^2\alpha_{11} \text{ for } s = 35; \\ 3\alpha_{47} + x_{12}^3\alpha_{11} \text{ for } s = 47. \end{cases}$$

Proof. Given a subset $I \subseteq e(G,p)$ and a function $r : I \to \mathbb{Z}^+$ denote by $y_I^{r(I)} \in H^*(G/T;\mathbb{F}_p)$ the monomial $\prod_{t\in I} y_t^{r(t)}$, where $\mathbb{Z}^+$ is the set of all positive integers. We call $y_I^{r(I)}$ *p–monotonous* if $r(t) < k_t$ for all $t \in I$ ([DZ$_1$, §5]).

Let $\Phi_{G,\mathbb{F}_p} = \{\gamma_s\}_{s\in r(G,p)}$, $\deg\gamma_s = 2s$ be the set of p–primary polynomials on G ([DZ$_1$, Definition 4]). In the context of [DZ$_1$, §6] each $\gamma_s \in \Phi_{G,\mathbb{F}_p}$ can be presented as

$$(4.2)\quad \gamma_s = \beta_s + \sum \beta_{I,r} y_I^{r(I)} \text{ with } \beta_s, \beta_{I,r} \in \ker\psi_p^*,$$

where the sum is over all *p–monotonous* $y_I^{r(I)}$ with

$$\deg(y_I^{r(I)}) = 2(r_1 i_1 + \cdots + r_t i_t) \leq 2s.$$

Applying the operator φ in [DZ$_2$; (2.7)] to (4.2) yields in $E_3^{2s-2,1}(G;\mathbb{F}_p)$ the relation

(4.3) $\xi_{2s-1} = [\varphi(\gamma_s)] = \mathcal{D}(\beta_s) + \sum x_I^{r(I)} \mathcal{D}(\beta_{I,r})$,

where the first equality comes from the definition of the class ξ_{2s-1} [DZ$_2$; Definition 2.3], the second is obtained by comparing the definitions of φ in [DZ$_2$; (2.7)] with $\mathcal{D}$ in (2.4), and where $\mathcal{D}(\beta_s)$, $\mathcal{D}(\beta_{I,r}) \in \mathcal{T}$ by (3.6).

Assume that $\deg \beta_{I,r} = c$. By i) of Lemma 1 β_s, $\beta_{I,r} \in \ker \psi_p^*$ implies that

(4.4) $\beta_s = b_s\theta_s + \tau_s$, $\beta_{I,r} = \begin{cases} \tau_c \text{ if } c \notin r(G,p) \\ b_{I,r}\theta_c + \tau_c \text{ if } c \in r(G,p) \end{cases}$,

where $b_s, b_{I,r} \in \mathbb{F}_p$, $\tau_c \in \langle \theta_s \rangle_{t \in r(G,p), t<c}$. Consequently

(4.5) $\mathcal{D}(\beta_s) = b_s\alpha_{2s-1}$; $\mathcal{D}(\beta_{I,r}) = \begin{cases} 0 \text{ if } c \notin r(G,p) \\ b_{I,r}\alpha_{2c-1} \text{ if } c \in r(G,p) \end{cases}$.

by Lemma 3. Substituting (4.5) in (4.3) we get the desired expression (4.1) of ξ_{2s-1} in terms of α_{2c-1}'s.

Finally, we remark that, in the context of [DZ$_1$], all the polynomials γ_s have been concreted presented in the form of (4.2) (as examples, see in [DZ$_1$; (6.2), (6.3)] for the cases $G = E_7$ and $p = 2, 3$) and the method in §5.3 to compute $b_{s,t}$ in (2.4) are applicable to determine b_s and $b_{I,r}$ in (4.5). This explains the algorithm obtaining the relations in Theorem 2.□

4.2. Proof of Theorem 1. The presentations of $H^*(G;\mathbb{F}_p)$ in Theorem 1 come from i) of Lemma 4, together the degree set $r(G,p)$ given in the table in §2. It should be noticed that, in a characteristic $p \neq 2$, the factor $\Delta(\alpha_{2s-1})_{s \in r(G,p)}$ in Lemma 4 can be replaced by the exterior algebra $\Lambda(\alpha_{2s-1})_{s \in r(G,p)}$ because odd dimensional cohomology classes are all square free.

According to ii) of Lemma 4, results on $\mathcal{P}^k(\alpha_{2s-1})$ are verified by Lemma 2. It remains to decide $\delta_p(\alpha_{2s-1})$.

It was shown in [DZ$_2$; (3.10)] that, with respect to the inclusion $e(G,p) \subset r(G,p)$ (see iii) of Lemma 1), one has

$$\delta_p(\xi_{2s-1}) = \begin{cases} -x_{2s} \text{ if } s \in e(G,p); \\ 0 \text{ if } s \notin e(G,p). \end{cases}$$

Applying δ_p to the expressions of ξ_{2s-1} in Theorem 2 then verifies the results on $\delta_p(\alpha_{2s-1})$ in Theorem 1.□

4.3. Applications: the algebra $H^*(G;\mathbb{F}_2)$. It follows from the proof of Lemma 4 that the set of 2–transgressive generators on $H^*(G;\mathbb{F}_2)$ is unique. Moreover, one can deduce from (1.1) of Theorem 1 the next result, that expresses the ring $H^*(G;\mathbb{F}_2)$ solely by these generators (without resorting to the 2–special Schubert classes on G/T).

Corollary 1. *With respect to the 2–transgressive generators on $H^*(G;\mathbb{F}_2)$ one has the isomorphisms of algebras*

$$H^*(G_2;\mathbb{F}_2) = \mathbb{F}_2[\alpha_3]/\left\langle \alpha_3^4 \right\rangle \otimes \Lambda_{\mathbb{F}_2}(\alpha_5);$$

$$H^*(F_4;\mathbb{F}_2) = \mathbb{F}_2[\alpha_3]/\left\langle \alpha_3^4 \right\rangle \otimes \Lambda_{\mathbb{F}_2}(\alpha_5, \alpha_{15}, \alpha_{23});$$

$$H^*(E_6;\mathbb{F}_2)=\mathbb{F}_2[\alpha_3]/\left\langle \alpha_3^4\right\rangle \otimes \Lambda_{\mathbb{F}_2}(\alpha_5,\alpha_9,\alpha_{15},\alpha_{17},\alpha_{23});$$
$$H^*(E_7;\mathbb{F}_2)=\frac{\mathbb{F}_2[\alpha_3,\alpha_5,\alpha_9]}{\left\langle \alpha_3^4,\alpha_5^4,\alpha_9^4\right\rangle}\otimes \Lambda_{\mathbb{F}_2}(\alpha_{15},\alpha_{17},\alpha_{23},\alpha_{27});$$
$$H^*(E_8;\mathbb{F}_2)=\frac{\mathbb{F}_2[\alpha_3,\alpha_5,\alpha_9,\alpha_{15}]}{\left\langle \alpha_3^{16},\alpha_5^8,\alpha_9^4,\alpha_{15}^4\right\rangle}\otimes \Lambda_{\mathbb{F}_2}(\alpha_{17},\alpha_{23},\alpha_{27},\alpha_{29}).$$

Proof. In view of (1.1) it suffices to show that

$$(4.6)\quad \alpha_{2s-1}^2=\begin{cases} x_6 \text{ for } s=2 \text{ and in } G_2,F_4,E_6,E_7,E_8; \\ x_{4s-2} \text{ for } s=3,5 \text{ and in } E_7,\ E_8; \\ x_{30}+x_6^2x_{18} \text{ for } s=15 \text{ and in } E_8, \end{cases}$$

and that

(4.7) $\alpha_{2s-1}^2=0$ for those α_{2s-1} belonging to the exterior part.

These can be deduced directly from $\alpha_{2s-1}^2=\delta_2\mathcal{P}^{s-2}\alpha_{2s-1}$ and (1.1), together with the Adem relation [A] and the fact $\mathcal{P}^{s-2}\alpha_{2s-1}\in\mathcal{T}$ by Lemma 4.□

Remark 1. The rings $H^*(G;\mathbb{F}_2)$ (together with the $\mathcal{P}^k$ action on $H^*(G;\mathbb{F}_2)$) were first obtained by Borel, Araki, Shikata and Thomas [B, Ar, AS, T] which, in terms of generator and relations, agree with those given in Corollary 1. However, in these classical results there is no indication on the effect of Sq^1 action on the generators in the *exterior* part. The formulae for $\delta_2(\alpha_{2s-1})$ in (1.1) of Theorem 1 implies that these actions are highly nontrivial:

$$Sq^1(\alpha_{15})=\alpha_3^2\alpha_5^2,\ Sq^1(\alpha_{27})=\alpha_5^2\alpha_9^2 \text{ in } E_7,E_8;$$
$$Sq^1(\alpha_{23})=\alpha_3^2\alpha_9^2 \text{ in } E_7;$$
$$Sq^1(\alpha_{23})=\alpha_3^2\alpha_9^2+\alpha_3^8;\ Sq^1(\alpha_{29})=\alpha_{15}^2 \text{ in } E_8.\square$$

Traditionally, the cohomologies $H^*(G;\mathbb{F}_p)$ for exceptional G were calculated case by case, presented using generators from quite different origins (this happened, even for the case $p=2$, see [B; A; AS; T, Ko$_1$; KN]), and without referring to the integral cohomology $H^*(G)$. As a result one could hardly analyzing $H^*(G)$ from information about $H^*(G;\mathbb{F}_p)$. In comparison, since the *primary generators* on $H^*(G;\mathbb{F})$ ([DZ$_2$, Definition 2.3]) in various coefficients $\mathbb{F}$ stemming solely from Schubert presentation of the ring $H^*(G/T)$, the relationship between $H^*(G)$ and $H^*(G;\mathbb{F}_p)$ (for all prime p) are transparent with respect to these generators (see [DZ$_2$; Lemma 2.5; Lemma 3.3]). It is for this reason we are more interested in the presentation of the $\mathcal{A}_p$–module $H^*(G;\mathbb{F}_p)$ by the p–primary generators.

In [DZ$_2$; Theorem 1] $H^*(G;\mathbb{F}_2)$ is presented by the set $\left\{\xi_{2s-1}\right\}_{s\in r(G,p)}$ of p–primary generators as

$$H^*(G;\mathbb{F}_2)=\mathbb{F}_2[x_{2t}]/\left\langle x_{2t}^{k_t}\right\rangle_{t\in e(G,2)}\otimes\Delta(\xi_{2s-1})_{s\in r(G,2)}.$$

To specify the ring structure of $H^*(G;\mathbb{F}_2)$ with respect to $\left\{\xi_{2s-1}\right\}_{s\in r(G,2)}$ it suffices to find the expressions of all the squares ξ_{2s-1}^2 in the above presentation. This has been done in views of i) of Theorem 2, (4.6) and (4.7).

Corollary 2. *With respect to the 2–primary generators on $H^*(G;\mathbb{F}_2)$, one has the isomorphisms of algebras*

$$H^*(G_2;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle \otimes \Delta_{\mathbb{F}_2}(\xi_3) \otimes \Lambda_{\mathbb{F}_2}(\xi_5);$$
$$H^*(F_4;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle \otimes \Delta_{\mathbb{F}_2}(\xi_3) \otimes \Lambda_{\mathbb{F}_2}(\xi_5,\xi_{15},\xi_{23});$$
$$H^*(E_6;\mathbb{F}_2) = \mathbb{F}_2[x_6]/\left\langle x_6^2\right\rangle \otimes \Delta_{\mathbb{F}_2}(\xi_3) \otimes \Lambda_{\mathbb{F}_2}(\xi_5,\xi_9,\xi_{15},\xi_{17},\xi_{23});$$
$$H^*(E_7;\mathbb{F}_2) = \frac{\mathbb{F}_2[x_6,x_{10},x_{18}]}{\left\langle x_6^2,x_{10}^2,x_{18}^2\right\rangle} \otimes \Delta_{\mathbb{F}_2}(\xi_3,\xi_5,\xi_9) \otimes \Lambda_{\mathbb{F}_2}(\xi_{15},\xi_{17},\xi_{23},\xi_{27});$$
$$H^*(E_8;\mathbb{F}_2) = \frac{\mathbb{F}_2[x_6,x_{10},x_{18},x_{30}]}{\left\langle x_6^8,x_{10}^4,x_{18}^2,x_{30}^2\right\rangle} \otimes \Delta_{\mathbb{F}_2}(\xi_3,\xi_5,\xi_9,\xi_{15},\xi_{23}) \otimes \Lambda_{\mathbb{F}_2}(\xi_{17},\xi_{27},\xi_{29}),$$

where

$$\xi_3^2 = x_6 \text{ in } G_2,F_4,E_6,E_7,E_8,$$
$$\xi_5^2 = x_{10},\ \xi_9^2 = x_{18} \text{ in } E_7,E_8,$$
$$\xi_{15}^2 = x_{30};\ \xi_{23}^2 = x_6^6 x_{10} \text{ in } E_8.\square$$

Remark 2. Corollary 2 was applied in [DZ$_2$; §6] to determine the integral cohomology ring $H^*(G)$ with respect to the *integral primary generators*.$\square$
Remark 3. In [DZ$_2$; Theorems 3–5] the rings $H^*(G;\mathbb{F}_p)$ were presented by p–primary generators. Combining Theorems 1 and 2 with the Cartan formula [SE] determines the $\mathcal{A}_p$ action on $H^*(G;\mathbb{F}_p)$ with respect to these generators.$\square$

5 Proof of Lemma 2

In §5.1 we obtain formulae for the $\mathcal{P}^k$ action on the universal Chern classes of complex vector bundles. In §5.2 we present, for each exceptional G and prime $p = 2,3,5$, a set $\{\theta_s\}_{s\in r(G,p)}$ of generating polynomials for the ideal $\ker\psi_p^*$ in terms of Chern classes of certain vector bundle on BT. With these preliminaries Lemma 2 is establishes in §5.3.

5.1. The $\bmod p$–**Wu formulae.** Let $U(n)$ be the unitary group of rank n, and let $BU(n)$ be its classifying space. It is well known that, for a prime p,

$$H^*(BU(n),\mathbb{F}_p) = \mathbb{F}_p[c_1,\ldots,c_n]$$

where $1+c_1+\cdots+c_n \in H^*(BU(n),\mathbb{F}_p)$ is the *total Chern class* of the universal complex n–bundle ξ_n on $BU(n)$. This implies that each $\mathcal{P}^k c_m$ can be written as a polynomial in the $c_1,\ldots,c_n$, and such expression may be called the *mod p–Wu formula* for $\mathcal{P}^k c_m$ [P, Sh]. In the next result we present such formulae for certain $\mathcal{P}^k c_m$ that are barely sufficient for a proof of Lemma 2.

Proposition 1. *The following relations hold in $H^*(BU(n),\mathbb{F}_p)$*

i) $p = 2:$
$\mathcal{P}^r c_m = \sum\limits_{0\le t\le r} \binom{r-m}{t} c_{r-t}c_{m+t}$, where $\binom{n}{i} = n(n-1)\cdots(n-i+1)/i!$.

ii) $p = 3:$
$$\mathcal{P}^1 c_m = (m+2)c_{m+2} - c_1c_{m+1} + (c_1^2+c_2)c_m;$$
$$\mathcal{P}^2 c_m = c_2^2c_m + c_1c_3c_m - c_4c_m - c_1c_2c_{1+m} + (m+1)c_1^2c_{2+m}$$
$$+(m-1)c_2c_{2+m} - (m+1)c_1c_{3+m} + \tfrac{1}{2}(m^2+3m+2)c_{4+m};$$

$$\begin{aligned}\mathcal{P}^3c_m &= c_3^2c_m + c_2c_4c_m - c_1c_5c_m + c_6c_m - c_2c_3c_{1+m} + c_5c_{1+m}\\ &+mc_2^2c_{2+m} + (1+m)c_1c_3c_{2+m} - (1+m)c_4c_{2+m} - mc_1c_2c_{3+m}\\ &-c_3c_{3+m} + \tfrac{1}{2}(m^2+m)c_1^2c_{4+m} - m^2c_2c_{4+m} - \tfrac{1}{2}(m^2+m)c_1c_{5+m}\\ &+\tfrac{1}{6}(m^3+3m^2+2m-6)c_{6+m}\end{aligned}$$

iii) $p = 5$:

$$\begin{aligned}\mathcal{P}^1c_m &= (m+4)c_{m+4} - c_1c_{m+3} + (c_1^2 - 2c_2)c_{m+2}\\ &+(-c_1^3 - 2c_1c_2 + 2c_3)c_{m+1} + (c_1^4 + c_1^2c_2 + 2c_2^2 - c_1c_3 + c_4)c_m.\end{aligned}$$

Proof. For $p = 2$ the expansion of $\mathcal{P}^rc_m$ comes from the classical Wu–formula [W] as $c_r \bmod 2$ is the $2r^{th}$ Stiefel–Whitney class of the real reduction of ξ_n.

For $p > 2$ we have the general expansion of $\mathcal{P}^kc_m$ in terms of the Schur symmetric functions s_λ by the formula in [Du, (1.2)]

(5.1) $\mathcal{P}^k(c_m) \equiv \sum\limits_{\lambda} K^{-1}_{(1^{m-k},p^k),\lambda}s_\lambda \bmod p,$

where $K^{-1}_{(1^{m-k},p^k),\lambda}$ is the *inverse Kostka number* associated to the pair $\{\mu = (1^{m-k},\ p^k);\ \lambda\}$ of partitions, and where the sum is over all partitions λ of $m + 2k(p-1)$. We note in (5.1) that

(5.2) for those (p,k) concerned by Proposition 1, [ER, Corollary 2] and [Du, Corollary 5] can be applied to evaluate the coefficients $K^{-1}_{(1^{m-k},p^k),\lambda}$;

(5.3) each Schur function s_λ can be expanded as a polynomial in the c_r's by the classical Giambelli formula $s_\lambda = \det(c_{\lambda'_j+j-i})$ [M, p.36], where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the partition conjugate to λ.

Combining (5.2) and (5.3) one obtains the relations in the Proposition.□

Remark 4. We record below the presentation of (5.1) from which the relevant inverse Kostka numbers are transparent. For $p = 3$ we have

$$\begin{aligned}\mathcal{P}^1c_m &= ms_{(1^{m+2})} + s_{(1^{m-1},3)} - s_{(1^m,2)} - s_{(1^{m-2},2^2)};\\ \mathcal{P}^2c_m &= s_{(1^{m-2},3^2)} + (m-1)s_{(1^{m+1},3)} - s_{(1^{m-1},2,3)} - s_{(1^{m-3},2^2,3)}\\ &+\tfrac{m(m-1)}{2}s_{(1^{m+4})} - (m-1)s_{(1^{m+2},2)} - (m-2)s_{(1^m,2^2)} + 2s_{(1^{m-2},2^3)}\\ &+s_{(1^{m-4},2^4)};\\ \mathcal{P}^3c_m &= s_{(1^{m-3},3^3)} + (m-2)s_{(1^m,3^2)} - s_{(1^{m-2},2,3^2)} - s_{(1^{m-4},2^2,3^2)}\\ &+\tfrac{(m-1)(m-2)}{2}s_{(1^{m+3},3)} - (m-2)s_{(1^{m+1},2,3)} - (m-3)s_{(1^{m-1},2^2,3)}\\ &+2s_{(1^{m-3},2^3,3)} + s_{(1^{m-5},2^4,3)} + \tfrac{m(m-1)(m-2)}{6}s_{(1^{m+6})} - \tfrac{(m-1)(m-2)}{2}s_{(1^{m+4},2)}\\ &-\tfrac{(m-2)(m-3)}{2}s_{(1^{m+2},2^2)} + (2m-5)s_{(1^m,2^3)} + (m-5)s_{(1^{m-2},2^4)}\\ &-3s_{(1^{m-4},2^5)} - s_{(1^{m-6},2^6)}\end{aligned}$$

For $p = 5$ we have

$$\begin{aligned}\mathcal{P}^1c_m &= ms_{(1^{m+4})} + s_{(1^{m-1},5)} - s_{(1^m,4)} - s_{(1^{m-2},2,4)} + s_{(1^{m+1},3)}\\ &+s_{(1^{m-1},2,3)} + s_{(1^{m-3},2^2,3)} - s_{(1^{m+3},2)} - s_{(1^m,2^2)} - s_{(1^{m-2},2^3)} - s_{(1^{m-4},2^4)}.\square\end{aligned}$$

5.2. Generating polynomials for $\ker\psi_p^*$. For n indeterminacies $t_1,\cdots,t_n$ of degree 2 we set

(5.4) $1+e_1+\cdots+e_n=\prod_{1\leq i\leq n}(1+t_i)$,

That is, e_i is the i^{th} elementary symmetric functions in $t_1,\cdots,t_n$ with degree $2i$. For an exceptional G with rank n, assume that the set $\{\omega_i\}_{1\leq i\leq n}\subset H^2(BT)$ of fundamental weights (cf. Lemma 1) is so ordered as the vertices in the Dynkin diagram of G in [Hu, p.58]. We introduce a set of polynomials $c_k(G)\in H^{2k}(BT)$ in $\omega_1,\cdots,\omega_n$ for $G=F_4,E_6,E_7,E_8$.

Definition 2. If $G=F_4$ we let $c_k(F_4)$, $1\leq k\leq 6$, be the polynomial obtained from $e_k(t_1,\cdots,t_6)$ in (5.4) by letting

$$t_1=\omega_4;\quad t_2=\omega_3-\omega_4;\quad t_3=\omega_2-\omega_3;$$
$$t_4=\omega_1-\omega_2+\omega_3;\quad t_5=\omega_1-\omega_3+\omega_4;\quad t_6=\omega_1-\omega_4.$$

If $G=E_n$, $n=6,7,8$, we let $c_k(E_n)$, $1\leq k\leq n$, be the polynomial obtained from $e_k(t_1,\cdots,t_n)$ in (5.4) by letting

$$t_1=\omega_n;\quad t_2=\omega_{n-1}-\omega_n;\quad \cdots;$$
$$t_{n-3}=\omega_4-\omega_5;\quad t_{n-2}=\omega_3-\omega_4+\omega_2;$$
$$t_{n-1}=\omega_1-\omega_3+\omega_2;\quad t_n=-\omega_1+\omega_2.\square$$

We emphasis at this stage that

Lemma 5. *The class $1+c_1(F_4)+\cdots+c_6(F_4)\in H^*(BT)$ (resp. $1+c_1(E_n)+\cdots+c_n(E_n)\in H^*(BT)$, $n=6,7,8$) is the total Chern class of a 6–dimensional (resp. n–dimensional) complex bundle ξ_G on BT.*

Moreover, $c_1(G)\in H^2(BT)$ can be expressed in terms of weights as

$$c_1(G)=\begin{cases} 3\omega_1 \text{ for } G=F_4; \\ 3\omega_2 \text{ for } G=E_6,E_7,E_8. \end{cases}$$

Proof. For a 2–dimensional cohomology class $t\in H^2(BT)$ let L_t be the complex line bundle on BT with Euler class t. Then

$$\xi_{F_4}=\bigoplus_{1\leq i\leq 6}L_{t_i}\ (\text{resp. } \xi_{E_n}=\bigoplus_{1\leq i\leq n}L_{t_i},\ n=6,7,8),$$

where t_i is the linear form in the weights given in Definition 2.

The expressions of all $c_r(G)$ by the special Schubert classes on G/T were deduced in [DZ$_1$; Lemma 4], by which the formula for $c_1(G)$ is a special case.$\square$

Let (G,p) be a pair with $H^*(G)$ containing non–trivial p–torsion. In Propositions 2–4 we present, in accordance to $p=2,3,5$, a set $\{\theta_s\}_{s\in r(G,p)}\subset H^*(BT;\mathbb{F}_p)$ $=\mathbb{F}_p[\omega_1,\cdots,\omega_n]$ of generating polynomial for $\ker\psi_p^*$ (derived from the set of p–primary polynomials on G [DZ$_1$; Definition 4] by the method illustrated in the proof of [DZ$_1$; Proposition 3])

Proposition 2. *For $G=G_2$, F_4 and E_8, a set $\{\theta_i\}_{i\in r(G,2)}$ of generating polynomials for $\ker\psi_2^*$ is given by*

$\{\theta_i\}_{i\in r(G,2)}$	G_2	F_4	E_8
θ_2	$\omega_1^2+\omega_1\omega_2+\omega_2^2$	c_2	c_2
θ_3	ω_2^3	c_3	c_3
θ_5			$c_5+\omega_2 c_4$
θ_8		$c_4^2+\omega_1^2 c_6$	$c_8+c_4^2+\omega_2^2 c_6+\omega_2^3 c_5+\omega_2^8$
θ_9			$\omega_2^2 c_7+\omega_2 c_8+\omega_2^3 c_6$
θ_{12}		$c_6^2+c_4^3$	$c_6^2+c_4^3$
θ_{14}			$c_7^2+c_4^2 c_6+\omega_2^2 c_6^2$
θ_{15}			$c_7 c_8+\ \omega_2^7 c_8+\omega_2^3 c_4 c_8$

and for $G=E_6,E_7$ *by*

$$\{\theta_i\}_{i\in r(E_6,2)}=\{\theta_i\mid_{c_7=c_8=0}\}_{i\in r(E_8,2)\setminus\{14,15\}};$$
$$\{\theta_i\}_{i\in r(E_7,2)}=\{\theta_i\mid_{c_8=0}\}_{i\in r(E_8,2)\setminus\{15\}}.\square$$

Proposition 3. *For an exceptional* G *with* $G\neq G_2$*, a set* $\{\theta_i\}_{i\in r(G,3)}$ *of generating polynomials for* $\ker\psi_3^*$ *is given by*

$\{\theta_i\}$	F_4	E_6	E_7
θ_2	$\omega_1^2-c_2$	$\omega_2^2-c_2$	$\omega_2^2-c_2$
θ_4	$c_2^2-c_4$	$c_2^2-c_4$	$c_2^2-c_4$
θ_5		$c_5+c_2c_3$	
θ_6	$c_2c_4-c_6$	$c_2c_4+c_3^2-c_6$	$-\omega_2^3c_3+c_2c_4-\omega_2c_5\ +c_3^2-c_6$
θ_8	$-c_2c_6$	$-c_4^2$	$-c_4^2+c_2c_3^2\ -\omega_2c_7+c_3c_5$
θ_9		c_6c_3	
θ_{10}			$-c_4c_3^2+c_2c_3c_5+\ c_3c_7-c_5^2$
θ_{14}			$c_4c_5^2+c_2c_5c_7+\ c_7^2$
θ_{18}			$c_2c_3^3c_7+c_3^6+\ c_3^2c_5c_7+c_3c_5^3$

$\{\theta_i\}$	E_8
θ_2	$\omega_2^2-c_2$
θ_4	$c_2^2-c_4$
θ_8	$-\omega_2^5c_3-\omega_2^3c_5\ -\omega_2^2c_3^2-\omega_2^2c_6-\omega_2c_7+\ c_3c_5$
θ_{10}	$-c_4c_3^2+c_2c_3c_5+\ c_2c_8+c_3c_7-c_5^2$
θ_{14}	$c_4c_3c_7+\omega_2^3c_3c_8+\ c_2c_3^2c_6+c_2c_5c_7\ -\omega_2c_5c_8-c_3^2c_8+c_3c_5c_6+c_7^2$
θ_{18}	$-c_2c_4^4+c_4c_3^2c_8+c_4c_6c_8-c_4c_7^2\ -c_2c_3^3c_7-c_2c_3c_5c_8+\ c_2c_3c_6c_7$ $-\omega_2c_3c_6c_8-c_3^6\ -c_3^2c_6^2-c_5c_6c_7+c_6^3$
θ_{20}	$-c_2c_3c_7c_8+\ \omega_2c_3c_8^2+c_3^2c_6c_8+c_5c_7c_8$
θ_{24}	$c_8^3+c_2c_3^2c_8^2-\omega_2c_3c_6^2c_8+c_2c_3c_5c_6c_8-c_3^2c_5^2c_8-\omega_2c_3c_5c_7c_8-c_3c_7^3$ $-\omega_2c_3c_6c_7^2-c_2c_3c_5c_7^2+c_5^2c_7^2+c_2c_4^2c_7^2-c_5c_6^2c_7-c_3^2c_5c_6c_7+c_3^4c_5c_7$ $-c_2c_5^3c_7-c_3^2c_6^3\ +c_2c_4c_6^3+c_3^4c_6^2$

Proposition 4. *For* $G=E_8$*, a set of generating polynomials for* $\ker\psi_5^*$ *is given by*

$\theta_2=-\omega_2^2-c_2;$
$\theta_6=2\omega_2^6-2\omega_2^3c_3-2\omega_2c_5\ -2c_3^2-c_6;$
$\theta_8=-\omega_2^8-\omega_2^4c_4\ -2\omega_2^3c_5-\omega_2c_3c_4-\omega_2c_7-c_3c_5\ -c_4^2-c_8;$
$\theta_{12}=-2\omega_2^4c_4^2-\omega_2^4c_8+\omega_2^3c_3^3+2\omega_2^3c_4c_5-2\omega_2^2c_3^2c_4-\omega_2^2c_3c_7-2\omega_2c_3c_4^2$
$\quad+c_3^4-c_3c_4c_5-2c_5c_7+2c_6^2;$

$$
\begin{aligned}
\theta_{14} = & -2\omega_2^{10}c_4 + 2\omega_2^8 c_3^2 - 2\omega_2^7 c_7 + \omega_2^5 c_3 c_6 - 2\omega_2^4 c_3 c_7 + 2\omega_2^4 c_5^2 + \omega_2^3 c_3^2 c_5 \\
& +\omega_2^3 c_4 c_7 + \omega_2 c_3 c_4 c_6 - \omega_2 c_4^2 c_5 + \omega_2 c_5 c_8 - 2\omega_2 c_6 c_7 + c_3^2 c_4^2 - c_3^2 c_8 \\
& +2c_3 c_4 c_7 + c_4^2 c_6 + c_4 c_5^2 + c_7^2; \\
\theta_{18} = & -2\omega_2^8 c_5^2 + 2\omega_2^7 c_3^2 c_5 - 2\omega_2^6 c_3^2 c_6 + \omega_2^6 c_3 c_4 c_5 + 2\omega_2^5 c_3^2 c_7 + 2\omega_2^4 c_3^2 c_8 \\
& +\omega_2^4 c_4 c_5^2 + 2\omega_2^3 c_3 c_4^3 - \omega_2^3 c_3 c_5 c_7 + 2\omega_2^3 c_4^2 c_7 - 2\omega_2^3 c_5^3 - \omega_2^2 c_3^4 c_4 - 2\omega_2^2 c_3^3 c_7 \\
& +\omega_2^2 c_3 c_4^2 c_5 + 2\omega_2^2 c_4^4 - \omega_2^2 c_4^2 c_8 - \omega_2 c_3^4 c_5 - 2\omega_2 c_3 c_7^2 + \omega_2 c_4^3 c_5 - 2\omega_2 c_4 c_5 c_8 \\
& +\omega_2 c_5^2 c_7 - c_3^2 c_4 c_8 + c_3^2 c_5 c_7 - 2c_3 c_4^2 c_7 + 2c_3 c_4 c_5 c_6 - c_3 c_5^3 - 2c_3 c_7 c_8 + c_4 c_7^2; \\
\theta_{20} = & -\omega_2^{17}c_3 - \omega_2^{13}c_7 + 2\omega_2^{12}c_4^2 + 2\omega_2^{12}c_8 + 2\omega_2^{11}c_3 c_6 + \omega_2^{10}c_3^2 c_4 - \omega_2^9 c_4 c_7 \\
& +2\omega_2^8 c_4^3 - \omega_2^7 c_3 c_5^2 - \omega_2^6 c_3^3 c_5 - \omega_2^6 c_3^2 c_8 + \omega_2^6 c_4 c_5^2 - 2\omega_2^5 c_3^5 + \omega_2^5 c_3 c_4^3 \\
& + \omega_2^5 c_4^2 c_7 + 2\omega_2^5 c_5^3 - \omega_2^4 c_3^4 c_4 - 2\omega_2^4 c_3 c_4^2 c_5 - 2\omega_2^4 c_4 c_5 c_7 + \omega_2^3 c_3^4 c_5 \\
& -2\omega_2^3 c_3^2 c_4 c_7 - \omega_2^3 c_3 c_4 c_5^2 + \omega_2^2 c_3^6 + 2\omega_2^2 c_3^2 c_4^3 - \omega_2^2 c_3^2 c_5 c_7 - 2\omega_2 c_3^5 c_4 \\
& +2\omega_2 c_3^3 c_5^2 + 2\omega_2 c_3^2 c_6 c_7 + \omega_2 c_4 c_5^3 + 2c_3^4 c_8 + c_3^3 c_4 c_7 + c_3^2 c_7^2 + 2c_3 c_4^3 c_5 \\
& +2c_4^5 + c_4^3 c_8 - 2c_5^4; \\
\theta_{24} = & -\omega_2^{16}c_8 - \omega_2^{13}c_3 c_8 - 2\omega_2^9 c_3 c_4 c_8 + 2\omega_2^7 c_4 c_5 c_8 + \omega_2^6 c_4 c_6 c_8 - 2\omega_2^6 c_5^2 c_8 \\
& +2\omega_2^5 c_3 c_8^2 + \omega_2^5 c_4 c_7 c_8 - \omega_2^5 c_5 c_6 c_8 + 2\omega_2^4 c_4 c_8^2 - \omega_2^4 c_5 c_7 c_8 + \omega_2^3 c_3^3 c_4 c_8 \\
& -2\omega_2^3 c_3^2 c_7 c_8 + \omega_2^3 c_3 c_4 c_6 c_8 - 2\omega_2^3 c_3 c_5^2 c_8 + \omega_2^3 c_6 c_7 c_8 + \omega_2^2 c_4 c_5^2 c_8 - \omega_2^2 c_6 c_8^2 \\
& -2\omega_2 c_3 c_4 c_8^2 - \omega_2 c_4 c_5 c_6 c_8 - 2\omega_2 c_7 c_8^2 + c_3^4 c_4 c_8 + 2c_3 c_5 c_8^2 + c_3 c_6 c_7 c_8 \\
& -2c_5^2 c_6 c_8. \square
\end{aligned}
$$

5.3. Proof of Lemma 2. Let (G,p) be a pair with G exceptional and $H^*(G)$ containing non–trivial p–torsion. Granted with the concrete expressions of the set $\{\theta_s\}_{s\in r(G,p)}$ of generating polynomials for $\ker \psi_p^*$ in §5.2 and the mod p Wu–formulae in §5.1, we complete the proof of Theorem 1 by showing Lemma 2.

If $(G,p) = (G_2, 2)$, Lemma 2 is directly shown by the computation (see in Proposition 2 for the expressions of θ_2, θ_3 in G_2)

$$\mathcal{P}^1\theta_2 = \mathcal{P}^1(\omega_1^2 + \omega_1\omega_2 + \omega_2^2) = \omega_1^2\omega_2 + \omega_1\omega_2^2 = \theta_3 + \omega_1\theta_2.$$

So we can assume from now on that $G \neq G_2$.

Let $\mathbb{F}_p[G]$ be the subring of $H^*(BT;\mathbb{F}_p)$ generated by $c_i = c_i(G)$ and the weight ω_r with $r = 1$ for F_4, $r = 2$ for E_6, E_7, E_8. Then $\{\theta_i\}_{i\in r(G,p)} \subset \mathbb{F}_p[G]$ by Proposition 2–4. Since the $c_r(G)$'s are the mod p reduction of the Chern classes of a vector bundle on BT, the Wu–formulae in Proposition 1, together with the Cartan–formula [SE], are applicable to express each $\mathcal{P}^k\theta_r$ as an element in $\mathbb{F}_p[G]$. It remains to sort out the number $b_{s,t} \in \mathbb{F}_p$ in the equation (2.4).

The expressions of $\mathcal{P}^k\theta_r \in \mathbb{F}_p[G]$ may appear lengthy (in particular, this happens when $G = E_8$ and $p = 3$ and 5). However, we have two practical methods implementing $b_{s,t} \in \mathbb{F}_p$. The first utilizes *Mathematica*, while the second lifts the computation to an appropriate S^1–bundle on BT at where, θ_r and $\mathcal{P}^k c_m$ admit much simpler expressions.

Proof of Lemma 2 (Method I). Based on certain build-in functions of *Mathematica* the procedure to compute $b_{s,t}$ in (2.4) is given as follows.

For an $i \in r(G,p)$ denote by $\mathcal{G}_i(G,p) \subset \mathbb{F}_p[G]$ a Gröbner basis of the ideal generated by the subset $\{\theta_j\}_{j\in r(G,p), j<i}$. Let $\{s,t\} \subset r(G,p)$ be a pair with $t = s + k(p-1)$.

Step 1. Call GroebnerBasis[,] to compute $\mathcal{G}_t(G,p)$;

Step 2. Call PolynomialReduce[, ,] to compute the residue h_a of $P^k\theta_s - a\theta_t$ module $\mathcal{G}_t(G,p)$, $a \in \mathbb{F}_p$;

Step 3. Take $b_{s,t} = \{a \in \mathbb{F}_p \mid h_a = 0\}$.$\square$

To demonstrate the second method a few notations are required. Let $\kappa : S(BT) \to BT$ be the oriented S^1–bundle on BT with Euler class $\omega_r \in H^2(BT)$, where $r = 1$ for F_4 and $r = 2$ for E_6, E_7, E_8. Then we have

$$H^*(S(BT);\mathbb{F}_p) = H^*(BT;\mathbb{F}_p) \mid_{\omega_r=0}$$

and the induced ring map κ^* on cohomology is given simply by $\kappa^*\theta = \theta \mid_{\omega_r=0}$.

Example. Let $\{\theta_i\}_{i\in r(G,p)}$ be the set of generating polynomials for $\ker \psi_p^*$. Then $\kappa^*\theta_i$ has simpler expression than that of θ_i. As an example consider the case $(G,p) = (E_8, 5)$. We get from Proposition 4 that

$$\begin{aligned}
\kappa^*\theta_2 &= -c_2\\
\kappa^*\theta_6 &= -c_6 - 2c_3^2;\\
\kappa^*\theta_8 &= -c_8 - c_3c_5 - c_4^2;\\
\kappa^*\theta_{12} &= -2c_5c_7 + 2c_6^2 - c_3c_4c_5 + c_3^4;\\
\kappa^*\theta_{14} &= -c_3^2c_8 + c_7^2 + 2c_3c_4c_7 + c_4^2c_6 + c_4c_5^2 + c_3^2c_4^2;\\
\kappa^*\theta_{18} &= -2c_3c_7c_8 - c_3^2c_4c_8 + c_4c_7^2 + c_3^2c_5c_7 - 2c_3c_4^2c_7 + 2c_3c_4c_5c_6 - c_3c_5^3;\\
\kappa^*\theta_{20} &= c_4^3c_8 + 2c_3^4c_8 + c_3^2c_7^2 + c_3^3c_4c_7 - 2c_5^4 + 2c_3c_4^3c_5 + 2c_4^5;\\
\kappa^*\theta_{24} &= 2c_3c_5c_8^2 + c_3c_6c_7c_8 - 2c_5^2c_6c_8 + c_3^4c_4c_8.
\end{aligned}$$

Moreover, on the subring $\kappa^*\mathbb{F}_5[E_8] = \mathbb{F}_5[c_2, \cdots, c_8]$, one has

$$\mathcal{P}^1 c_m = (m+4)c_{m+4} - 2c_2c_{m+2} + 2c_3c_{m+1} + (2c_2^2 + c_4)c_m$$

by Proposition 1, where we have reserved c_r for κ^*c_r, and where $\kappa^*c_1 = 0$ by Lemma 5.□

The second proof of Lemma 2 may appear elaborate, but is useful in confirming the results obtained from the first one, and may be free of computer.

Proof of Lemma 2 (Method II). The proof is divided into two cases in accordance with $\kappa^*\theta_t = 0$ and $\kappa^*\theta_t \neq 0$.
Case 1. $\kappa^*\theta_t = 0$. This happens precisely when $p = 2, t = 9$ and $G = E_6, E_7, E_8$ by Proposition 2–4. Direct computation shows that

$$\begin{aligned}
\mathcal{P}^1\theta_8 &= \theta_9 + \omega_2^4\theta_5\\
\mathcal{P}^4\theta_5 &= \theta_9 + c_4\theta_5 + (\omega_2^2c_4 + c_6)\theta_3 + (\omega_2^2c_5 + c_7)\theta_2.
\end{aligned}$$

These verify the assertions $b_{5,9} = b_{8,9} = 1$ in Lemma 2.
Case 2. $\kappa^*\theta_t \neq 0$. Applying κ^* to the relation (2.4) we get in $H^*(S(BT);\mathbb{F}_p)$ that

(5.5) $\mathcal{P}^k(\kappa^*\theta_s) = b_{s,t}\kappa^*\theta_t + \tau_t$ with $\tau_t \in \langle\kappa^*\theta_s\rangle_{s\in r(G,p),s<t}$.

Computation in the case $(G,p) = (E_8, 5)$ is typical enough of the remaining cases. Carrying on the discussion in the Example we find that

$$
\begin{aligned}
&\mathcal{P}^1\kappa^*\theta_2 = \kappa^*\theta_6 + (c_4 - 2c_2^2)\kappa^*\theta_2;\\
&\mathcal{P}^1\kappa^*\theta_8 = \kappa^*\theta_{12} + (-c_2^2 + 2c_4)\kappa^*\theta_8 + (-2c_3^2 + 2c_6)\kappa^*\theta_6\\
&\qquad +(2c_2c_8 + 2c_3c_7 - c_4c_6 + 2c_5^2)\kappa^*\theta_2\\
&\mathcal{P}^1\kappa^*\theta_{14} = \kappa^*\theta_{18} - (c_2^2c_3^2 + c_2c_8 + c_3^2c_4 + 2c_3c_7 - c_4c_6 + 2c_5^2)\kappa^*\theta_8\\
&\qquad +2c_4^3\kappa^*\theta_6 - (c_2c_3^3c_5 - c_2c_3^2c_4^2 + 2c_2c_3c_4c_7 + c_2c_4^2c_6 + c_2c_4c_5^2 - c_2c_7^2\\
&\qquad +c_3^2c_4c_6 + c_3c_4^2c_5 + c_3c_6c_7 - c_4^2c_8 + 2c_4c_5c_7 + c_4c_6^2 - 2c_5^2c_6 + c_8^2)\kappa^*\theta_2\\
&\mathcal{P}^1\kappa^*\theta_{20} = \kappa^*\theta_{24} + c_6\kappa^*\theta_{18} + (c_3^2c_4 - c_3c_7 - c_4c_6 + 2c_5^2)\kappa^*\theta_{14}\\
&\qquad -(-c_3c_4c_5 + c_4^3 - 2c_4c_8 + c_5c_7)\kappa^*\theta_{12}\\
&\qquad -(c_2c_4^2c_6 + 2c_3c_5c_8 + c_4^2c_8 - c_4c_5c_7 + c_4c_6^2)\kappa^*\theta_8\\
&\qquad -(c_2^2c_7^2 + c_2c_3c_6c_7 + 2c_3^3c_4c_5 + c_3^2c_4^3 + 2c_3^2c_5c_7 - c_3c_4c_5c_6 - c_3c_7c_8\\
&\qquad +c_4^2c_5^2 - 2c_5^2c_8 + 2c_5c_6c_7)\kappa^*\theta_6\\
&\qquad -(-2c_2c_4^3c_8 - c_2c_5^4 + c_2c_6c_7^2 - c_3^3c_5c_8 - c_3^2c_4c_5c_7 + c_3c_4^3c_7 - c_3c_4^2c_5c_6\\
&\qquad +c_3c_5c_7^2 + c_3c_6^2c_7 + c_4^4c_6 + c_4^3c_5^2 + c_5^3c_7)\kappa^*\theta_2
\end{aligned}
$$

These imply that $b_{s,s+4} = 1$ for $s = 2, 8, 14, 20$ by (5.5).□

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